- There are 4 hours available for the problems.
- Every problem is worth 10 points.
- Be clear when using a theorem. When you are using an obscure theorem, cite a source.
- Use a different sheet for each exercise.
- Clearly write DRAFT on any draft page you hand in.

MOAWOA :: SOLUTIONS

20 March 2015

Problem 1. Let $n \geq 4$ be an integer. Suppose that in a group of $2n$ people everyone speaks at least one of ℓ languages. Suppose that each of the ℓ languages is spoken by at least k people. We want these people to stand in a circle in such a way that each two neighbors have a common language.

(a) If $\ell = 2$, what is the minimal value of k such that this is always possible?

(b) If $\ell = 3$, what is the minimal value of k such that this is always possible?

Proposed by Daniël Kroes.

Solution.

(a) Suppose the languages are Dutch and English. Consider the situation where n people speak Dutch and the other n people speak English. Then clearly they cannot stand in a circle such that the condition is met, so $k \geq n+1$. We claim that $k = n+1$, so assume that there are at least $n+1$ people that speak Dutch and $n+1$ people that speak English. Then among the at least $n+1$ people that speak Dutch, there must be at least 2 people who also speak English. Now we can ensure that everyone can speak with both its neighbours by making sure that everyone that speaks Dutch are on the same arc of the circle determined by these two persons, and that all persons that do not speak Dutch are on the other arc.

(b) Again, the answer is $k = n+1$. Suppose the languages are Dutch, English and French. To see that $k \leq n$ does not work, consider the situation where n people speak both Dutch and English, and the other n people speak French. Then they cannot stand in a circle such that the condition is met. Now suppose that each language is spoken by at least $n+1$ people. Analogously to above we want to find three different people such that one of them speaks both Dutch and English, another speaks both English and French and the third speaks both French and Dutch. Let a, b and c be the number of persons that speak Dutch and English, English and French and French and Dutch, respectively. W.l.o.g. we assume $a \geq b, c$. Analogously to part (a) we find $a, b, c \geq 2$. Suppose that $a = 2$ hence also $b = c = 2$. Then there are precisely $n+1$ people speaking each language (otherwise $b > 2$ or $c > 2$), and the $n-1$ people not speaking French must speak English (as $b = 2$) and Dutch (as $c = 2$) so there are at least $n - 1 \geq 3$ people speaking both Dutch and English, contradicting $a = 2$. Therefore, we have $a > 2$. Now, choose someone speaking both French and Dutch and someone else speaking both English and French, which is possible as $b \geq 2$. Now, as $a > 2$ there is someone different from these two speaking both Dutch and English, showing the required. □

Problem 2. Let $n > 1$ be an integer. Show that there exist positive integers a, b, c satisfying $a + b = n$ and $|ab - c^2| \leq 4$.

Proposed by Merlijn Staps.

Solution. It is easily checked that the statement holds for $n \leq 12$. Suppose $n \geq 13$ and write $n = 5k + \ell$ where k, ℓ are integers satisfying $k \geq 3$ and $|\ell| \leq 2$. Taking $a = 4k$, $b = k + \ell$ and $c = 2k + \ell$ (note that $k \geq 3$ implies $b, c > 0$) yields $a + b = 5k + \ell = n$ and $|ab - c^2| = |4k(k + \ell) - (2k + \ell)^2| = |\ell^2| = |\ell|^2 \leq 4$, as desired. □

Problem 3. Let $\mathbb{N} = \{1, 2, 3, \dots\}$ be the set of positive integers and let $f : \mathbb{N} \to \mathbb{N}$ be a bijective function.

(a) Is it possible that $\sum_{n=1}^{\infty} \frac{1}{nf(n)}$ diverges?

(b) Is it possible that $\sum_{n=1}^{\infty} \frac{1}{n+f(n)}$ converges?

Proposed by Daniël Kroes.

Solution.

(a) No, by the Cauchy-Schwarz inequality we find for all $N \in \mathbb{N}$ that

$$\sum_{n=1}^{N} \frac{1}{nf(n)} \leq \sqrt{\sum_{n=1}^{N} \frac{1}{n^2}} \cdot \sqrt{\sum_{n=1}^{N} \frac{1}{f(n)^2}} \leq \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^2}} \cdot \sqrt{\sum_{n=1}^{\infty} \frac{1}{f(n)^2}} = \sqrt{\frac{\pi^2}{6}} \cdot \sqrt{\frac{\pi^2}{6}} = \frac{\pi^2}{6}$$

which shows that $\sum_{n=1}^{\infty} \frac{1}{nf(n)}$ converges, with limit at most $\frac{\pi^2}{6}$.

(b) Yes, let $\{m_1, m_2, m_3, \dots, \} = \{3, 5, 6, 7, 9, \dots\}$ enumerate all integers that are not a power of 2. Define the function f by $f(m_i) = 2^{i-1}$ for all $i \in \mathbb{N}$ and $f(2^j) = m_{j+1}$ for all $j \geq 0$. Then clearly f is bijective. Then

$$\sum_{i=1}^{\infty} \frac{1}{m_i + f(m_i)} = \sum_{i=1}^{\infty} \frac{1}{m_i + 2^{i-1}} \leq \sum_{i=1}^{\infty} \frac{1}{2^{i-1}} = 2$$

and

$$\sum_{j=0}^{\infty} \frac{1}{2^j + f(2^j)} \leq \sum_{j=0}^{\infty} \frac{1}{2^j} = 2,$$

are both convergent, so the same holds for their sum, which is precisely $\sum_{n=1}^{\infty} \frac{1}{n+f(n)}$. □

Problem 4. Let G be a finite group with identity e and let H and K be subgroups of G such that $|H| \cdot |K| = |G|$ and $H \cap K = \{e\}$. Prove that $H' \cap K' = \{e\}$ for all conjugate subgroups H' and K' of H and K, respectively.
For a subgroup Y of a group X, a conjugate subgroup of Y is a subgroup of X that is of the form xYx^{-1} for some $x \in X$.

Proposed by Raymond van Bommel.

Solution. We claim that $G = \{hk : h \in H, k \in K\}$. Since $|H| \cdot |K| = |G|$ it suffices to show that all these elements are different. So let $h_1, h_2 \in H$ and $k_1, k_2 \in K$ satisfy $h_1k_1 = h_2k_2$ then $h_2^{-1}h_1 = k_2k_1^{-1}$. Since $h_2^{-1}h_1 \in H$ and $k_2k_1^{-1} \in K$ and $H \cap K = \{e\}$ we find $h_2^{-1}h_1 = e = k_2k_1^{-1}$ so indeed $h_1 = h_2$ and $k_1 = k_2$.
Now, by conjugating H' and K' by the same appropriate element we may assume without loss of generality that $H' = H$. Suppose that $K' = (hk)K(hk)^{-1} = h(kKk^{-1})h^{-1} = hKh^{-1}$ and let $g \in H \cap hKh^{-1}$. Then $g = hkh^{-1}$ for some $k \in K$, which rewrites as $k = h^{-1}gh \in H$ so $k \in K \cap H$, hence $k = e$. But this implies that $g = heh^{-1} = e$, as required. □

Alternative solution. Let $X = (G/H) \times (G/K)$ be the set of pairs of a right coset of H in G and a right coset of K in G. Let G act on X coordinatewise by left multiplication: $g \cdot (aH, bK) = (gaH, gbK)$. The stabilizer of $(H, K) \in X$ is $H \cap K = \{e\}$. By the orbit-stabilizer theorem, the orbit of $(H, K) \in X$ has size $|G| = |X|$. Hence, the action of G on X is free and transitive. In particular, for each $a, b \in G$, the stabilizer of $(aH, bK) \in X$, which equals $aHa^{-1} \cap bKb^{-1}$, is trivial. □

Problem 5. Let $n \geq 2$ be an integer and let $A = (a_{i,j})$ be a real $n \times n$ matrix with entries $a_{i,j}$ different from 0 that satisfy

$$a_{i,j}a_{i+1,j+1} - a_{i+1,j}a_{i,j+1} = ij$$

for all $i, j \in \{1, 2, \ldots, n-1\}$. Determine the rank of A.

Proposed by Christophe Debry.

Solution. The rank equals 2. For $n = 2$ the only imposed equation implies $\det(A) = 1$, so A is invertible and has rank 2. Let $n \geq 3$ and define for all $i, j \in \{1, 2, \ldots, n\}$ and $k \leq \min(n-i, n-j)$ the matrix

$$M_{i,j}^{(k)} = \begin{pmatrix} a_{i,j} & a_{i,j+1} & \cdots & a_{i,j+k} \\ a_{i+1,j} & a_{i+1,j+1} & \cdots & a_{i+1,j+k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i+k,j} & a_{i+k,j+1} & \cdots & a_{i+k,j+k} \end{pmatrix}$$

Then $\det(M_{i,j}^{(1)}) = ij$ for all $i, j \in \{1, 2, \ldots, n-1\}$ and one can easily check that for all $i, j \in \{1, 2, \ldots, n-2\}$ we have

$$a_{i+1,j+1} \det(M_{i,j}^{(2)}) = \begin{vmatrix} \det(M_{i,j}^{(1)}) & \det(M_{i,j+1}^{(1)}) \\ \det(M_{i,j+1}^{(1)}) & \det(M_{i+1,j+1}^{(1)}) \end{vmatrix} = \begin{vmatrix} ij & i(j+1) \\ (i+1)j & (i+1)(j+1) \end{vmatrix} = 0,$$

so $\det(M_{i,j}^{(2)}) = 0$, since $a_{i+1,j+1} \neq 0$.
We now prove that the third row of A is a linear combination of the first two rows of A. Because $\det(M_{1,1}^{(2)}) = 0$ but $\det(M_{1,1})^{(1)} \neq 0$ there exist $c_1, c_2 \in \mathbb{R}$ with $a_{3,i} = c_1a_{1,i} + c_2a_{2,i}$ for all $i \in \{1, 2, 3\}$. Analogously, there exist $c_1', c_2' \in \mathbb{R}$ with $a_{3,i} = c_1'a_{1,i} + c_2'a_{2,i}$ for $i \in \{2, 3, 4\}$ (consider $M_{1,2}^{(2)}$ and $M_{1,2}^{(1)}$). We now know

$$c_1(a_{1,2}, a_{1,3}) + c_2(a_{2,2}, a_{2,3}) = c_1'(a_{1,2}, a_{1,3}) + c_2'(a_{2,2}, a_{2,3}).$$

Because $\det(M_{1,2})^{(1)} \neq 0$ we find that $(a_{1,2}, a_{1,3})$ and $(a_{2,2}, a_{2,3})$ form a basis of $\mathbb{R}^2$ over $\mathbb{R}$ so $c_1 = c_1'$ and $c_2 = c_2'$. By an inductive argument, $c_1a_{1,i}+c_2a_{2,i} = a_{3,i}$ holds for all $i \in \{1, 2, \ldots, n\}$. Analogously, any row in A is a linear combination of the two preceding rows, so by induction each row is a linear combination of the first rows, showing that A has rank at most 2. Since $\det(M_{1,1}^{(1)} \neq 0$ we find that the rank is indeed exactly 2. □

Problem 6. A biologist studies an exceptional bacterial species. When a bacterium of this species takes d minutes to divide, its two descendants take d and $d+1$ minutes to divide. The biologist starts with a single bacterium that takes 1 minute to divide.
Show that when the total number of bacteria becomes even for the n-th time, it stays even for exactly n minutes.

Proposed by Merlijn Staps.

Solution. For $n \geq 0$ define A_n to be the total number of bacteria after n minutes. Write $B_n = A_n - A_{n-1}$ for the number of bacteria that divides exactly n minutes after the start of the experiment. Such a bacterium can be described by a sequence $(a_1, \ldots, a_r)$ of natural numbers that indicates how long the predecessors of this bacterium took to divide. We then must have $a_1 + \cdots + a_r = n$ and $a_i \in \{a_{i-1}, 1 + a_{i-1}\}$ for $1 < i \leq r$. Note that each sequence corresponds to a partition of n and that B_n counts the number of such partitions with the property that if b occurs in the partition, $b-1$ must occur as well. These partitions are dual to the partitions that consist of distinct elements, so B_n also counts the number of partitions of n in distinct elements. We will show that B_n is odd precisely when n is a pentagonal number, i.e. precisely when n is of the form $\frac{k(3k\pm1)}{2}$ for some natural number k. Since the difference between the 2ℓ-th and the $(2\ell-1)$-th element in the set $\{1, 2, 5, 7, 12, 15, \ldots\}$ of pentagonal numbers is equal to ℓ this is sufficient for the desired conclusion.
A partition λ of n in distinct parts corresponds bijectively with a sequence $(a_1, \ldots, a_r)$ of natural numbers with sum n that satisfies $a_1 > \cdots > a_r$. We define $p_\lambda = \max\{i : a_i = a_1 + 1 - i\}$ and $q_\lambda = a_r$. To λ we now associate another partition λ' of n in distinct parts in the following way. If $p_\lambda < q_\lambda$ we subtract 1 of the first p_λ parts of the partition and we add a new element p_λ. If $p_\lambda \geq q_\lambda$ we remove the last part and we add 1 to the first p_λ elements. This yields a second partition λ' that satisfies $\lambda'' = \lambda$, unless λ has the form $(2\ell, 2\ell-1, \ldots, \ell+1)$ (we then have $(p, q) = (\ell, \ell+1)$ and the part that we add will not be smaller than the last part) or the form $(2\ell-1, 2\ell-2, \ldots, \ell)$ (we then have $(p, q) = (\ell, \ell)$ and a problem arises because we add 1 to the ℓ-th element which we already removed; the resulting partition λ' does not satisfy $\lambda'' = \lambda$). If these cases do not occur, we can divide the partitions of n into distinct parts in pairs (λ, λ') and therefore the total number B_n of such partitions will be even. If such a case does occur, there is exactly one partition that does not belong to a pair. In that case B_n is odd and this happens precisely when n is of the form

$$2\ell + (2\ell-1) + \cdots + (\ell+1) = \frac{\ell(3\ell+1)}{2}$$

or of the form

$$(2\ell-1) + (2\ell-2) + \cdots + \ell = \frac{\ell(3\ell-1)}{2}.$$

In conclusion, B_n is odd if and only if n is a pentagonal number. □

Remark. The argument in the second part of the proof can be extended to prove Euler's theorem on pentagonal numbers. This theorem states that

$$\prod_{k\geq1}(1-x^k) = 1 + \sum_{k\geq1}(-1)^k \left(x^{k(3k+1)/2} + x^{k(3k-1)/2}\right).$$

The left-hand side of this equation is the generating function for the difference between the number of partitions of n in distinct odd parts and the number of partitions of n in distinct even parts. One can show that this difference is ±1 if n is a pentagonal number and 0 otherwise. This also means that that B_n is odd exactly for pentagonal numbers n.